CLUB of MATHEMATICS
IISER Thiruvananthapuram

Epiphany: The CMIT Mathematics Tournament

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Solutions to Problems

Refer to the problem set for instructions and point distribution.

Section A

1. If G is a finite group which contains exactly 24 elements of order 6, then the number of cyclic subgroups of G having order 6 is (a) 6 (b) 12 (c) 18 (d) 24.

   Answer: (b) 12.

   Solution: We know that every cyclic group of order 6 has $\phi(6) = 2$ generators. Note that if $H = \langle a \rangle$ is a subgroup of order 6, then $O(a) = 6$. Thus, any cyclic subgroup of order 6 contains 2 elements of order 6 and any element of order 6 is in precisely one subgroup of that order (namely the one it generates). Therefore the 24 elements of order 6 should be evenly divided across subgroups and there are exactly $\frac{24}{2} = 12$ cyclic subgroups of order 6.

   Contributed by: Sagnik S. ; *Reference*: [1]

2. Let A be 5×5 matrix with real entries such that the sum of the entries in each row of A is 1. Then the sum of all entries in A^3 is (a) 3 (b) 15 (c) 5 (d) 125.

   Answer: (c) 5.

   Solution: Note that if $A = I$, it is trivial. Assume that $A \neq I$, then note that corresponding to the eigenvalue 1,

   $$e = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   is an eigenvector for the matrix A. $\therefore$ $Ae = e$. Thus sum of all entries in A^3 is

   $$e^T A^3 e = e^T A^2 (Ae) = e^T A^2 e = e^T A e = e^T e = 5$$

   Contributed by: Anand C.

3. For complex values of x if

   $$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = L$$

   then (a) $L = 0$ (b) $L = 1$ (c) $\Im(L) > 0$ (d) the limit L doesn't exist.

   Remark: $\cos(iz) = \cosh z$, $\sin(iz) = i \sinh z$, $\Im(z) =$ *imaginary part of* z.

   Answer: (d) the limit L doesn't exist.

   Solution (sketch): Let $x = a + ib$ where $(a, b) \in \mathbb{R}^2$. Thus $x \to 0 \implies (a, b) \to (0, 0)$.

   $$\therefore L = \lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = \lim_{(a,b) \to (0,0)} (a + ib) \sin\left(\frac{a - ib}{a^2 + b^2}\right)$$

. Expanding, we obtain

$$L = \lim_{(a,b)\to(0,0)} [(af(a,b) + bg(a,b)) + i(bf(a,b) - ag(a,b))]$$

where

$$f(a,b) = \frac{1+e^{-\frac{2b}{a^2+b^2}}}{2e^{-\frac{b}{a^2+b^2}}} \sin\left(\frac{a}{a^2+b^2}\right) \; ; \; g(a,b) = \frac{1-e^{-\frac{2b}{a^2+b^2}}}{2e^{-\frac{b}{a^2+b^2}}} \cos\left(\frac{a}{a^2+b^2}\right)$$

From the above form of L we can conclude that the limit does not exist since its value will depend on $(a,\, b)$ path in complex space.

Remark: Note that for real values of x, the limit exists and is equal to 0.

Constructed by: Sagnik S.

4. Let f be a function defined on $\mathbb{Z}^+$ by $f(1) = 1$, $f(2n) = 2f(n)$, $f(2n+1) = 4f(n)\ \forall\ n \in \mathbb{N}$. Then the number of solutions to $f(n) = 8$ is (a) 1 (b) 5 (c) 3 (d) 9.

   Answer: (c) 3.

   Solution: First we show by induction that $f(n) \geq n\ \forall\ n \in \mathbb{N}$. Then we check that for $1 \leq n \leq 8$, there are only three solutions for $f(n) = 8$, viz. $n = 5, 6, 8$.

   Contributed by: Anand C.

5. Given that, for the set $S \subset \mathbb{C}$, if $z \in S$, then

   $$|z-1| = |z-i| = |z+1|$$

   On the complex plane, the set S geometrically represents a (a) triangle (b) pair of straight lines (c) finite number of points (d) ellipse.

   Answer: (c) finite number of points.

   Solution: Assume $|z-1| = |z-i| = |z+1| = r$. Note that these equations successively represent a circle of radius r centering points $(1,0)$, $(0,i)$ and $(-1,0)$. Since three distinct intersecting circles can only have a single common point, the number of solutions for z is exactly one (and precisely $z = 0$).

   Contributed by: Sagnik S.

6. If $P(x)$ is a real valued non-constant polynomial then

   $$\lim_{k\to\infty} \frac{P(k+1)}{P(k)}$$

   equals to (a) 1 (b) 0 (c) -1 (d) the leading coefficient of $P(x)$ (e) doesn't always converge.

   Answer: (a) 1.

   Solution:

$$\begin{aligned}
\lim_{k\to\infty} \frac{P(k+1)}{P(k)} &= \lim_{k\to\infty} \frac{\sum_{r=0}^{n} a_r (k+1)^r}{\sum_{r=0}^{n} a_r k^r} \\
&= \lim_{k\to\infty} \sum_{s=0}^{n} \frac{a_s (k+1)^s}{\sum_{r=0}^{n} a_r k^r} \\
&= \lim_{k\to\infty} \sum_{s=0}^{n} \frac{a_s}{a_n k^{n-s} (\frac{1}{1+\frac{1}{k}})^s + \cdots + a_s (\frac{1}{1+\frac{1}{k}})^s + \frac{a_{s-1}}{k} (\frac{1}{1+\frac{1}{k}})^s + \cdots + \frac{a_0}{k^s} (\frac{1}{1+\frac{1}{k}})^s} \\
&= \frac{a_n}{a_n} = 1.
\end{aligned}$$

   Contributed by: Anand C. ; *Solution by*: Sagnik S.

7. The series
$$\sum_{n=0}^{\infty}(n+1)(x)^n$$
(a) always diverges (b) is always bounded (c) always converges to $(1+x)^{-2}$
(d) converges to $(1+x)^{-2}$ only when $|x|<1$
(e) converges to $(1-x)^{-2}$ only when $|x|<1$
(f) converges to $x(1-x)^{-2}$ only when $|x|<1$.

Answer: (e) converges to $(1-x)^{-2}$ only when $|x|<1$.

Solution: Note that,
$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{d}{dx}\sum_{n=0}^{\infty}x^{n+1}$$
The geometric series $\sum_{n=0}^{\infty}x^n$ is finite only when $|x|<1$, and it converges to $\frac{x}{1-x}$. Thus,
$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{d}{dx}(\frac{x}{1-x}) = \frac{1}{(1-x)^2}, \text{ only when } |x|<1.$$

Contributed by: Sagnik S. ; *Reference*: [2]

8. If Morty has an infinite number of green, cyan, yellow and violet Pickle Ricks in a vessel, the minimum number of Pickel Ricks Morty must take out of the vessel to guarantee he has a pair of the same colored Pickle Ricks is (a) 2 (b) 4 (c) 5 (d) 6.

Answer: (c) 5.

Solution: Morty must pull 5 Pickle Ricks out of the drawer to guarantee he has a pair of the same colored Pickle Ricks. We are using the *Pigeonhole Principle*. In this case the pigeons are the Pickle Ricks he pulls out and the holes are the colors. Thus, if he pulls out 5 Pickle Ricks, the Pigeonhole Principle assures that two of them have will the same color. Also, note that it is possible to pull out 4 Pickle Ricks without obtaining a pair of same color.

Contributed by: Anand C. and Sagnik S. ; *Reference*: [3]

9. If $n \equiv 1 \pmod 2$ and $n \geq 3$, then the number of perfect squares mod 2^n is (a) $\frac{2^{n-1}+5}{3}$ (b) $\frac{2^{n-1}+4}{3}$ (c) $\frac{2^{n-1}+5}{4}$ (d) $2^{n-1}+5$.

Answer: (a) $\frac{2^{n-1}+5}{3}$.

Solution: Given that $n \equiv 1 \pmod 2 \implies n$ is an odd positive number. Let $s(n)$ denote the number of squares mod n and $q(n)$ the number of quadratic residues mod n. Note that it is known to us that $q(2^n) = 2^{n-3}$. Also for $n=3$, $s(2^3)=3$. We assume that the formula $s(2^n) = \frac{2^{n-1}+5}{3}$ holds true for all odd $n \leq 2k$. We show that it holds for $n = 2k+1$ as well. We first trivially obtain the formula $s(2^n) = q(2^n) + s(2^{n-2})$. Thus,
$$s(2^{2k+1}) = q(2^{2k+1}) + s(2^{(2k+1)-2}) = 2^{(2k+1)-3} + \frac{2^{[(2k+1)-2]-1}+5}{3} = \frac{2^{n-1}+5}{3}.$$

Contributed by: Sagnik S. ; *Reference*: [4]

10. Let $n \geq 3$ be an integer. Assume that inside a big circle, exactly n small circles of radius r can be drawn so that each small circle touches the big circle and also touches both its adjacent small circles. Then, the radius of the big circle is (a) r cosec$(\frac{\pi}{n})$ (b) r $(1+\text{cosec}\frac{2\pi}{n})$ (c) $r(1+\text{cosec}\frac{\pi}{2n})$ (d) $r(1+\text{cosec}\frac{\pi}{n})$.

Answer: (d) $r(1+\text{cosec}\frac{\pi}{n})$.

Solution (sketch): Let s be the distance between the centre of the big circle and the centre of (any) one of the small circles. Then there exists a right triangle with hypotenuse s, side r and one angle $\frac{\pi}{n}$. The other side of the same triangle is a part of the radius of the big circle. We can thus stare at the diagram and find that the radius of the big circle is $r(1+\text{cosec}\frac{\pi}{n})$.

Contributed by: Sagnik S. ; *Reference*: [5]

Section B

1. Prove that every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.

   Remark: The Fibonacci sequence is a sequence of numbers $\{F_n\}_{n=1}^{\infty}$ defined by the linear recurrence equation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$.

   Solution: Let $P(n)$ be the statement that "n can be expressed as the sum of distinct terms in the Fibonacci sequence". Note that $P(1)$ is trivially true. Let $P(n)$ be true for all $n \leq k$, we want to show that $P(k+1)$ is also true, and we will be done.
   Now for $P(k+1)$, there are two cases:

   Case 1: $k+1$ is itself a Fibonacci number. Then we are done.

   Case 2: $k+1$ is not a Fibonacci number. Then there exists a $m \in \mathbb{N}$ such that $F_m < k+1 < F_{m+1}$. Since $F_m < k+1$, we have $k+1 = F_m + (k+1) - F_m$. Now $(k+1) - F_m < k+1 \implies P((k+1) - F_m)$ is true by induction hypothesis. Let $(k+1) - F_m = \sum_{s=1}^{t} F_{i_s}$, where all F_j's are distinct. Since $(k+1) - F_m < F_m$, we have $F_{i_s} \neq F_m$ for any $1 \leq s \leq t$. Thus, $(k+1) = F_m + \sum_{s=1}^{t} F_{i_s}$ concludes that $P(k+1)$ is true.

   Contributed by: Joyentanuj D. ; *Reference*: [6]

2. Construct a C^{∞} function $f : \mathbb{R} \to \mathbb{R}$ such that the integral $\int_{-\infty}^{\infty} |f|$ is finite but the series $s_n = \sum_{k=0}^{n} f(k)$ diverges.
   Remark: A C^{∞} function is a function that is differentiable for all degrees of differentiation.

   Solution: Consider the function

   $$\phi(x) = \begin{cases} 2^x(x-n)+1 & x \in (n - \frac{1}{2^n}, n),\ n \in \mathbb{N} \\ 2^x(n-x)+1 & x \in (n, n + \frac{1}{2^n}),\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   Note that

   $$\int_{-\infty}^{\infty} |\phi(x)| dx = \sum_{k=0}^{\infty} \frac{1}{2^k} = 1 < \infty \ ; \ s_n = \sum_{k=0}^{n} \phi(k) = \sum_{k=0}^{n} 1, \text{ which diverges.}$$

   .
   Now we just have to make ϕ a C^{∞} function, without entirely changing the properties above. To do this, we construct the following function and convolute ϕ with it to make ϕ smooth:

   $$\psi_k(x) = \begin{cases} c_k \exp\left(|x|^2 - \frac{1}{2^{2k}}\right)^{-1} & |x| < \frac{1}{2^{2k}},\ k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   where $c_k = \frac{1}{2^k}$. Our final C^{∞} function f will therefore be:

   $$f(x) = \sum f_k \ ; \ f_k = \int_{-\infty}^{\infty} \phi(x-y)\psi_k(x) dy$$

   Remark: Partial points will be given for only constructing the function ϕ.

   Contributed by: Soham G. ; *Solution by*: Soham G. and Sagnik S. ; *Reference*: [7]

3. Find all odd positive integers $n > 1$ such that for any two coprime divisors a, b of n the number $(a+b-1)$ is also a divisor of n.

   Solution: Let p be the least prime factor of n. Assume that $\exists\ a > 1$ such that $n = p^m a$ and $(a, p) = 1$. Then $(p-1, a) = d = 1$ (otherwise consider a prime divisor $q > 1$ of d, $q \leq p - 1 < p$ and $q|a$, then $q|n$ contradicting the minimality of p). $\therefore (a, a+p-1) = 1$.

   Note that, $a|n$, $p|n \implies (a+p-1)|n$. Also notice that all the prime divisors of n except p divide a from definition. Since $(a+p-1)$ divides n and is coprime with a, no other prime but p divides it. So $(a+p-1) = p^b$ for some $m \geq b \in \mathbb{N}$.

   Also $a|n$, $p^b|n \implies (a+p^b-1) = (2a+p-2)|n$. We can prove $(2a+p-2)$ is coprime with a and so $(2a+p-2) = p^c \implies 2p^{b-1} - 1 = p^{c-1}$. It's not very hard to prove this equation has a solution only if $b = c = 1$. But then we get $(a+p-1) = p \implies a = 1$, contradicting with the definition of a. So such a can't exist and therefore $n = p^m$, (p odd) are the only possible solutions of odd $n \in \mathbb{Z}$.

   Contributed by: Sagnik S. ; *Solution by*: Adib H. ; *Reference*: [8]

Section C1

1. Can you continuously deform a 2-dimensional plane into a real line, or more mathematically, is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?
   Is $\mathbb{R}^2$ homeomorphic to $\mathbb{R}^3$? Does the previous argument work here?
   Can this be generalized? Is $\mathbb{R}^2$ homeomorphic to $\mathbb{R}^n$ for any $n > 2$?

   Remark: A bijective continuous function $f : X \to Y$ between two topological spaces is a homeomorphism if the inverse function f^{-1} is continuous.

   Solution (sketch): $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. For the sake of argument, assume they were. We know that connectedness is invariant under homeomorphism. If there would be a homeomorphism f from $\mathbb{R}^2$ to $\mathbb{R}$, we subtract a point from $\mathbb{R}^2$, for example, $(0,0)$ and $\mathbb{R}^2$ still stays connected. By the property of f, $f(\mathbb{R}^2 \setminus \{(0,0)\})$ is still connected, but $f(\mathbb{R}^2 \setminus \{(0,0)\})$ is precisely $\mathbb{R} \setminus \{a\}$ for some $a \in \mathbb{R}$, which is not connected.

   $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$ either. Here we would use the property that homeomorphic spaces have the same fundamental groups, and will arrive at a contradiction for the spaces $\mathbb{R}^2 \setminus \{(x,y)\}$ and $\mathbb{R}^3 \setminus \{(x,y,z)\}$.

   We can generalize this by the use of Brouwer's fixed point theorem.

   Contributed by: Adittya C.

2. For a non-negative integer r, prove the following combinatorial identity using any method, preferably through a combinatorial argument:

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k} = (-1)^{[\frac{r}{2}]}\binom{n}{[\frac{r}{2}]}\frac{1+(-1)^r}{2}$$

   where, $[x]$ denotes the greatest integer less than or equal to x.

   Solution (sketch): The above equation in question can be restated as:

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k} = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (\text{-}1)^{\frac{r}{2}}\binom{n}{\frac{r}{2}} & \text{if } r \text{ is even} \end{cases}$$

   We now expand the expressions $(1+x)^n(1-x)^n$ and $(1-x^2)^n$ and compare coefficients to achieve the desired identity.

   Contributed by: Joyentanuj D.

3. Determine whether the set of polynomials

$$P_1(x) = (x-1)(x-2)\cdots(x-n) - 1, n \geq 1$$

$$P_2(x) = (x-1)(x-2)\cdots(x-n) + 1, n \geq 5$$

   are reducible over $\mathbb{Z}$ or not.

   Remark: A polynomial f is said to be irreducible over a field $\mathbb{F}$ if f cannot be factored into product of polynomials all of which have degree lower than f. If f is not irreducible over $\mathbb{F}$ then we say that f is reducible over $\mathbb{F}$.

   Solution (sketch): Assume that $P_1(x)$ factorizes into $P_1(x) = Q_1(x)R_1(x)$. Note that

$$P_1(i) = -1 \ \forall\ i \in \{1,2,\ldots,n\} \implies Q_1(i) = -R_1(i) = \pm 1 \ \forall\ i \in \{1,2,\ldots,n\}$$

   So, $Q_1(x)+R_1(x)$ has at least n zeros, despite their leading coefficients having the same sign. This will create a contradiction.

   Assume that $P_2(x)$ factorizes into $P_2(x) = Q_2(x)R_2(x)$. Note that

$$P_1(i) = 1 \ \forall\ i \in \{1,2,\ldots,n\} \implies Q_1(i) = R_1(i) = \pm 1 \ \forall\ i \in \{1,2,\ldots,n\}$$

   So, $Q_1(x) - R_1(x)$ has at least n zeros. This tells us to deduce that $P_2(x) = [Q_2(x)]^2 \implies n = 2k$ where $k = deg\ Q_2(x)$. Also, there are k values in $\{1,2,\ldots,2k\}$ at which $P_2(x)$ is 1 and k values at which $P_2(x)$ is -1. For $k \geq 3$, WLOG, there will be some $u \geq 4$ such that $u-1 \mid P_2(u) - P_2(1) = -2$, contradiction.

   Contributed by: Ammu A.

Section C2

1. The *sequence of averages* of a real-valued sequence $\{x_n\}$ is defined by $a_n = \frac{\sum_{i=1}^{n} x_i}{n}$. If $\{x_n\}$ is a bounded sequence of real numbers, then show that

$$\liminf\ x_n \leq \liminf\ a_n \leq \limsup\ a_n \leq \limsup\ x_n$$

Also show that $\{x_n\} \to L \implies \{a_n\} \to L$ and justify whether the converse is true or not.

Remark: For a real-valued sequence $\{s_n\}$*, if* $S_N := \{s_n : n > N\}$*, then* $\limsup\ s_n = \lim_{N\to\infty} \sup S_N$*,* $\liminf\ s_n = \lim_{N\to\infty} \inf S_N$*. If* $\{s_n\}$ *is bounded, both the limits exist.*

Solution: *Lemma*: For $\epsilon > 0$, $s_k \geq (\limsup\ s_n + \epsilon)$ and $s_k \leq (\liminf\ s_n - \epsilon)$ for only finitely many natural numbers k.

For a given bounded sequence of real numbers x_n, fix $\epsilon > 0$. Define $l := \limsup\ x_n$ and consider the set $K := \{k \in \mathbb{N} \mid x_k \geq l + \epsilon\}$. Thus, by the *Lemma*, K is a finite set. Now consider the following two disjoint sets:

$$U_n := \{i \in \mathbb{N} \mid i \in K, i \leq n\}\ ;\ T_n := \{i \in \mathbb{N} \mid i \notin K, i \leq n\}$$

such that $U_n \bigcup T_n = \{1, 2, \ldots, n\}$. Corresponding to these two sets, we define two sequences $\{u_n\} := \sum_{i\in U_n} x_i$ and $\{t_n\} := \sum_{i\in T_n} x_i$. Observe that $a_n = \frac{u_n}{n} + \frac{t_n}{n}$. As K is a finite set, $\{u_n\}$ is eventually constant $\implies \frac{u_n}{n} \to 0$ as $n \to \infty$. For $i \notin K$, $x_k \leq l + \epsilon$.

$$\therefore t_n = \sum_{i\in T_n} x_i \leq n(l + \epsilon) \implies \frac{t_n}{n} \leq l + \epsilon\ \forall\ n \in \mathbb{N}$$

Thus,

$$\limsup\ a_n = \limsup\ (\frac{u_n}{n} + \frac{t_n}{n}) \leq \limsup\ \frac{u_n}{n} + \limsup\ \frac{t_n}{n} = 0 + l = \limsup\ x_n$$

Similarly, $\liminf\ x_n \leq \liminf\ a_n$. And we already know that $\liminf\ a_n \leq \limsup\ a_n$.
Now if $\{x_n\} \to L$, then

$$\liminf\ x_n = \limsup\ x_n = L \implies L \leq \liminf\ a_n \leq \limsup\ a_n \leq L \implies \{a_n\} \to L.$$

For disproving the converse, take $\{x_n\} = \{(-1)^n \mid n \in \mathbb{N}\}$.
Here, $\{a_n\} \to L$ but $\{x_n\}$ doesn't converge.

Contributed and solution by: Sharvari T.

2. You have coins $C_1, C_2, \ldots, C_m$. For each r, coin C_r is biased, so that when tossed, it has a probability of $1/(2r + 1)$ of falling in heads. If all the n coins are tossed, what is the probability that the number of tails is even? Express the answer as a function of n.

Solution: Note that, when tossed, the coin C_r has a probability of $(1-\frac{1}{2r+1}) = \frac{2r}{2r+1}$ of falling in tails. We first define the sequence $T_r = \frac{2r}{2r+1}$. For given n, let $P_n :=$ "probability that the number of tails is even when all the n coins are tossed". Thus, $P_0 = 1$, $P_1 = 1 - T_1 = \frac{1}{3}$. For every $n > 0$ we find P_n by first flipping the first (n–1) coins, getting even number of tails with probability P_{n-1} and odd number of tails with probability $(1 - P_{n-1})$. Then we flip the coin C_n to get an even number of tails among n coins with probability

$$P_n = (1 - T_n)P_{n-1} + (1 - P_{n-1})T_n = T_n + (1 - 2T_n)P_{n-1} = \frac{2n}{2n+1} + (\frac{1-2n}{2n+1})P_{n-1}$$

.
Solving the recurrence relation, we obtain:

$$P_n = \frac{2n + 1 + (-1)^n}{4n + 2}.$$

Contributed by: Ananthakrishna G. and Sagnik S. ; *Reference*: [9]

3. Let T be a linear operator on a vector space V (ie. $T : V \to V$) and λ be an eigenvalue for T. Let K_λ be the generalised eigenspace wrt. λ, ie. $K_\lambda := \{v \in V \mid (T - \lambda I)^p v = 0,\ p \in \mathbb{Z}^+\}$.
(a) Show that K_λ is a T-invariant subspace.
(b) For any eigenvalue $\mu \neq \lambda$, show that the function $(T - \mu I)$ restricted to K_λ is one-one.

Remark: Let W be a subspace of V and $T : V \to V$ be a linear operator. Then W is called T-invariant if $T(W) \subset W$.

Solution: (a) We have to show that $T(K_\lambda) \subset K_\lambda$. Let $x \in K_\lambda$, then we will show that $T(x) \in K_\lambda$. Since T commutes with $(T - \lambda I)^p$, we obtain $(T - \lambda I)^p T(x) = T(T - \lambda I)^p(x)$. Thus $(T - \lambda I)^p T(x) = T(0)$, since $x \in K_\lambda \implies T(x) \in K_\lambda$, since $T(0) = 0$.

(b) Let $\mu \neq \lambda$ be another eigenvalue. Assume $(T - \mu I)x = 0$ for some $0 \neq x \in K_\lambda$. $x \in K_\lambda \implies (T - \lambda I)^p x = 0$ for some $p \in \mathbb{Z}^+$. Let q denote the least integer such that $(T - \lambda I)^q x = 0$. Denote $y = (T - \lambda I)^{q-1} x \implies y \neq 0$ and $(T - \lambda I)y = 0 \implies T(y) = \lambda y$. Now $(T - \mu I)y = (T - \mu I)(T - \lambda I)^{q-1}(x) = (T - \lambda I)^{q-1}(T - \mu I)(x) = 0 \implies T(y) = \mu y = \lambda y \implies \mu y - \lambda y = 0 \implies (\mu - \lambda)y = 0 \implies \mu = \lambda$, which is a contradiction. Thus, $x = 0$. So, $(T - \mu I)$ restricted to K_λ is one-one.

Contributed by: Kalin K.

Section D

1. You visit the Island of Moai in search of a mathematical treasure that was supposedly hid by Pólya many years earlier and you came to know about recently. The treasure had no importance to the island dwellers so they can give away its location once you ask them. But after reaching the island, you find out from a magic sculpture signed by Pólya himself that says that at any point of time, the island is inhabited by equal number of truth-tellers and liars. In the island, everybody knows whether another person from the island is a truth-teller or a liar. Your aim now is to identify a truth-teller in the island to get the correct information about the treasure. You can only ask a person A about another person B whether B is a liar.

Let N be the smallest possible number of questions you need to ask in order to guarantee that you find a truth-teller. Show that no such N exists and that you can never find a truth-teller in finite amount of time to ask about the treasure.

Solution (sketch): Assume a condition where the Island of Moai is inhabited by only two people, Adam and Eve. Also assume that Adam is a truth-teller and Eve, a liar. You ask Adam if Eve is a liar, the answer is "yes". Next you ask Eve if Adam is a liar, the answer is also "yes". Now switch their characters, keeping your question set same. The answers will still be same. You will not be able distinguish a truth-teller. Now, change your questions, ask one if the other is a liar and the other if this one is a truth-teller. You will end up with the same dilemma as before. Now try to do this with more number of inhabitants on the island. You will never find a finite value for N.

Remarks: The author guesses this puzzle can be solved with a number of different techniques including Combinatorial Game Theory, Graph Theory, Induction, etc. Innovative solutions are welcome.

Constructed by: Sagnik S. ; *Reference*: [10]

References

[1] Joseph A. Gallian, Contemporary Abstract Algebra.

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