Epiphany: The CMIT Mathematics Tournament

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Read the instructions carefully.

Maximum time: 3 hours.

No electronic devices are to be used during the test.

The test is of 60 points for both **Category I** (Mathematics majors) and **Category II** (Foundation course students and other majors) students.

Point distribution: Section **A**: 10×2, Section **B**: 3×4, Section **C1/C2**: 3×6, Section **D**: 1×10.

Section **A**, **B** and **D** are to be attempted by both category candidates. However, it is *strictly* instructed that Category I students should only attempt section **C1** and Category II students should only attempt section **C2**.

Section A

1. If G is a finite group which contains exactly 24 elements of order 6, then the number of cyclic subgroups of G having order 6 is (a) 6 (b) 12 (c) 18 (d) 24.

2. Let A be 5×5 matrix with real entries such that the sum of the entries in each row of A is 1. Then the sum of all entries in A^3 is (a) 3 (b) 15 (c) 5 (d) 125.

3. For complex values of x if

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = L$$

   then (a) $L = 0$ (b) $L = 1$ (c) $\Im(L) > 0$ (d) the limit L doesn't exist.

   Remark: $\cos(iz) = \cosh z$, $\sin(iz) = i \sinh z$, $\Im(z) =$ *imaginary part of* z.

4. Let f be a function defined on $\mathbb{Z}^+$ by $f(1) = 1$, $f(2n) = 2f(n)$, $f(2n+1) = 4f(n)$ $\forall\, n \in \mathbb{N}$. Then the number of solutions to $f(n) = 8$ is (a) 1 (b) 5 (c) 3 (d) 9.

5. Given that, for the set $S \subset \mathbb{C}$, if $z \in S$, then

$$|z - 1| = |z - i| = |z + 1|$$

   On the complex plane, the set S geometrically represents a (a) triangle (b) pair of straight lines (c) finite number of points (d) ellipse.

6. If $P(x)$ is a real valued non-constant polynomial then

$$\lim_{k \to \infty} \frac{P(k+1)}{P(k)}$$

   equals to (a) 1 (b) 0 (c) -1 (d) the leading coefficient of $P(x)$ (e) doesn't always converge.

7. The series

$$\sum_{n=0}^{\infty}(n+1)(x)^n$$

(a) always diverges (b) is always bounded (c) always converges to $(1+x)^{-2}$
(d) converges to $(1+x)^{-2}$ only when $|x|<1$
(e) converges to $(1-x)^{-2}$ only when $|x|<1$
(f) converges to $x(1-x)^{-2}$ only when $|x|<1$.

8. If Morty has an infinite number of green, cyan, yellow and violet Pickle Ricks in a vessel, the minimum number of Pickel Ricks Morty must take out of the vessel to guarantee he has a pair of the same colored Pickle Ricks is (a) 2 (b) 4 (c) 5 (d) 6.

9. If $n \equiv 1 \pmod 2$ and $n \geq 3$, then the number of perfect squares mod 2^n is (a) $\frac{2^{n-1}+5}{3}$ (b) $\frac{2^{n-1}+4}{3}$ (c) $\frac{2^{n-1}+5}{4}$ (d) $2^{n-1}+5$.

10. Let $n \geq 3$ be an integer. Assume that inside a big circle, exactly n small circles of radius r can be drawn so that each small circle touches the big circle and also touches both its adjacent small circles. Then, the radius of the big circle is (a) r cosec$(\frac{\pi}{n})$ (b) r $(1+$ cosec$\frac{2\pi}{n})$ (c) $r(1+$ cosec$\frac{\pi}{2n})$ (d) $r(1+$ cosec$\frac{\pi}{n})$.

Section B

1. Prove that every positive integer n can be expressed as the sum of distinct terms in the Fibonacci sequence.

   Remark: The Fibonacci sequence is a sequence of numbers $\{F_n\}_{n=1}^{\infty}$ defined by the linear recurrence equation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$.

2. Construct a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ such that the integral $\int_{-\infty}^{\infty}|f|$ is finite but the series $s_n = \sum_{k=0}^{n} f(k)$ diverges.

   Remark: A C^∞ function is a function that is differentiable for all degrees of differentiation.

3. Find all odd positive integers $n > 1$ such that for any two coprime divisors a, b of n the number $(a+b-1)$ is also a divisor of n.

Section C1

1. Can you continuously deform a 2-dimensional plane into a real line, or more mathematically, is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?
   Is $\mathbb{R}^2$ homeomorphic to $\mathbb{R}^3$? Does the previous argument work here?
   Can this be generalized? Is $\mathbb{R}^2$ homeomorphic to $\mathbb{R}^n$ for any $n > 2$?

   Remark: A bijective continuous function $f : X \to Y$ between two topological spaces is a homeomorphism if the inverse function f^{-1} is continuous.

2. For a non-negative integer r, prove the following combinatorial identity using any method, preferably through a combinatorial argument:

$$\sum_{k=0}^{r}(-1)^k\binom{n}{k}\binom{n}{r-k} = (-1)^{[\frac{r}{2}]}\binom{n}{[\frac{r}{2}]}\frac{1+(-1)^r}{2}$$

   where, $[x]$ denotes the greatest integer less than or equal to x.

3. Determine whether the set of polynomials

$$P_1(x) = (x-1)(x-2)\cdots(x-n)-1, n \geq 1$$

$$P_2(x) = (x-1)(x-2)\cdots(x-n)+1, n \geq 5$$

   are reducible over $\mathbb{Z}$ or not.

   Remark: A polynomial f is said to be irreducible over a field $\mathbb{F}$ if f cannot be factored into product of polynomials all of which have degree lower than f. If f is not irreducible over $\mathbb{F}$ then we say that f is reducible over $\mathbb{F}$.

Section C2

1. The *sequence of averages* of a real-valued sequence $\{x_n\}$ is defined by $a_n = \frac{\sum_{i=1}^{n} x_i}{n}$. If $\{x_n\}$ is a bounded sequence of real numbers, then show that

$$\liminf\ x_n \leq \liminf\ a_n \leq \limsup\ a_n \leq \limsup\ x_n$$

Also show that $\{x_n\} \to L \implies \{a_n\} \to L$ and justify whether the converse is true or not.

Remark: For a real-valued sequence $\{s_n\}$*, if* $S_N := \{s_n : n > N\}$*, then* $\limsup\ s_n = \lim_{N\to\infty} \sup S_N$*,* $\liminf\ s_n = \lim_{N\to\infty} \inf S_N$*. If* $\{s_n\}$ *is bounded, both the limits exist.*

2. You have coins $C_1, C_2, \ldots, C_m$. For each r, coin C_r is biased, so that when tossed, it has a probability of $1/(2r+1)$ of falling in heads. If all the n coins are tossed, what is the probability that the number of tails is even? Express the answer as a function of n.

3. Let T be a linear operator on a vector space V (ie. $T : V \to V$) and λ be an eigenvalue for T. Let K_λ be the generalised eigenspace wrt. λ, ie. $K_\lambda := \{v \in V \mid (T - \lambda I)^p v = 0,\ p \in \mathbb{Z}^+\}$.
(a) Show that K_λ is a T-invariant subspace.
(b) For any eigenvalue $\mu \neq \lambda$, show that the function $(T - \mu I)$ restricted to K_λ is one-one.

Remark: Let W *be a subspace of* V *and* $T : V \to V$ *be a linear operator. Then* W *is called* T*-invariant if* $T(W) \subset W$*.*

Section D

1. You visit the Island of Moai in search of a mathematical treasure that was supposedly hid by Pólya many years earlier and you came to know about recently. The treasure had no importance to the island dwellers so they can give away its location once you ask them. But after reaching the island, you find out from a magic sculpture signed by Pólya himself that says that at any point of time, the island is inhabited by equal number of truth-tellers and liars. In the island, everybody knows whether another person from the island is a truth-teller or a liar. Your aim now is to identify a truth-teller in the island to get the correct information about the treasure. You can only ask a person A about another person B whether B is a liar.

Let N be the smallest possible number of questions you need to ask in order to guarantee that you find a truth-teller. Show that no such N exists and that you can never find a truth-teller in finite amount of time to ask about the treasure.

All the best!

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